THE LAMINAR BOUNDARY LAYER UNDER THE COAXIAL FLOW PAST A CYLINDER AND ON A CONTINUOUS CYLINDER

V. PECHOČ and J. KARNIŠ

Research Institute of Chemical Fibres, Svit

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Analytical solutions of the boundary layer on a continuous cylinder and in the flow past a cylinder for small values of the parameter $X = \nu x/UR^2$ are used to determine the error of the friction coefficient calculated by the approximate Kármán–Pohlhausen method. In case of the flow past a cylinder the earlier published results were confirmed and given precision; for the continuous cylinder it was found that the error of the approximate solution increases with X from 8% for $X \to 0$ to 8·7% for $X = 0·2$. According to Cebeci's calculations it is apparent that the error does not decrease even for larger X ($X \leq 250$) but remains between 8 and 9%.

The paper deals with the solution of the laminar boundary layer on a continuous cylinder. An example of such a cylinder may be for instance a fibre of constant radius moving at a steady speed between the feeder and the winding reel.

The problem of the boundary layer on a continuous moving flat surface and cylinder has been solved first by Sakiadis[1,2]. The solution on the cylinder was obtained by the approximate Kármán–Pohlhausen method[1] and analytically for the continuous flat surface[2]. Further papers concerning the continuous cylinder use either identical methods of solution in only formal modifications[3–5], or other approximate methods[6,7]. For this reason this work concentrated on accurate solution of the boundary layer equations on a continuous cylinder for small values of the curvature parameter X (Eq. (*6*)). Simultaneously, the published solutions for the boundary layer in the flow past a cylinder[8–13] were verified in the same region of the parameter X and given precision.

THEORETICAL

Differential equations for the laminar boundary layer in the flow past a cylinder and that on a continuous cylinder differ only in the boundary conditions. Their solutions, accordingly, will be obtained simultaneously.

Designating the coordinate in the direction of the axis of the cylinder by x and that perpendicular to the axis by r (its origin at the axis), the respective velocities in the direction of and perpendicular to the axis by u and v, the radius of the cylinder by R and the constant velocity of the bulk flow (flow past a cylinder) or the speed of the cylinder (continuous cylinder) by U, the laminar boundary layer may be described with commonly accepted simplifying assumptions by

$$r(\partial u/\partial x) + \partial(rv)/\partial r = 0\,, \tag{1}$$

$$u(\partial u/\partial x) + v(\partial u/\partial r) = \nu[(\partial^2 u/\partial r^2 + r^{-1}\,\partial u/\partial r)]\,, \tag{2}$$

with the boundary conditions

	flow past a cylinder	continuous cylinder	
$r = R$:	$u = v = 0$	$u = U\,;\quad v = 0$	(3)
$r \to \infty\,,\quad$ or	$u = U\,;\quad v = 0\,;$	$u = v = 0\,;\quad \partial u/\partial r = 0\,.$	(4)
$x = 0\,,\quad r > R$:	$\partial u/\partial r = 0$		

The familiar transformation, *i.e.* introducing the stream function as

$$\partial\psi/\partial x = -rv\,;\quad \partial\psi/\partial r = ru\,, \tag{5}$$

the dimensionless coordinates ξ and η, to replace x and r as

$$\xi = 4(\nu x/R^2 U)^{0\cdot5} = 4X^{0\cdot5}\,, \tag{6}$$

$$\eta = (U/\nu x)^{0\cdot5}\,[(r^2 - R^2)/4R] \tag{7}$$

and the dimensionless function $f(\xi, \eta)$ instead of the stream function ψ as

$$f(\xi, \eta) = \psi/(\nu x R^2 U)^{0\cdot5} \tag{8}$$

yields the following partial differential equation

$$\frac{\partial}{\partial\eta}\left[(1 + \xi\eta)\frac{\partial^2 f}{\partial\eta^2}\right] + f\frac{\partial^2 f}{\partial\eta^2} + \xi\left[\frac{\partial f}{\partial\xi}\frac{\partial^2 f}{\partial\eta^2} - \frac{\partial f}{\partial\eta}\frac{\partial^2 f}{\partial\xi\,\partial\eta}\right] = 0\,. \tag{9}$$

(A simplified notation f is used for the function $f(\xi, \eta)$ in the last equation for the sake of brevity).

The boundary conditions in the new coordinates are

	flow past a cylinder	continuous cylinder	
	$f(\xi, 0) = 0$,	$f(\xi, 0) = 0$,	(10)
	$(\partial f / \partial \eta)_{\eta=0} = 0$,	$(\partial f / \partial \eta)_{\eta=0} = 2$.	
	$(\partial f / \partial \eta)_{\eta \to \infty} = 2$,	$(\partial f / \partial \eta)_{\eta \to \infty} = 0$.	(11)

For solving Eq. (9) in region of small values of ξ one can use the power expansion of the function $f(\xi, \eta)$ as

$$f(\xi, \eta) = \sum_{i=0}^{\infty} \xi^i f_i(\eta) . \quad (12)$$

Taking only four terms of the last series which converges rapidly for small ξ four ordinary differential equations for the functions f_0 through f_3 result:

$$f_0''' + f_0 f_0'' = 0 , \quad (13)$$

$$f_1''' + f_0 f_1'' - f_0' f_1' + 2 f_0'' f_1 + \eta f_0''' + f_0'' = 0 , \quad (14)$$

$$f_2''' + f_0 f_2'' - 2 f_0' f_2' + 3 f_0'' f_2 + \eta f_1''' + f_1''(1 + 2 f_1) - (f_1')^2 = 0 , \quad (15)$$

$$f_3''' + f_0 f_3'' - 3 f_0' f_3' + 4 f_0'' f_3 + \eta f_2''' + f_2''(1 + 2 f_1) - 3 f_1' f_2' + 3 f_1'' f_2 = 0 , \quad (16)$$

with the boundary conditions

	flow past a cylinder	continuous cylinder	
$\eta = 0$:	$f_i = 0$	$f_i = 0$	(17)
	$f_i' = 0$	$f_0' = 2$; $f_1' = f_2' = f_3' = 0$	
$\eta \to \infty$:	$f_0' = 2$; $f_1' = f_2' = f_3' = 0$	$f_i' = 0$.	(18)

Eqs (13) through (16) have been solved by the Taylor expansion and the Runge–Kutta method taking by trial and error the value of $f_i''(0)$ so as to satisfy to a maximum accuracy possible with the used calculator HP 9100 A the boundary condition (18). The accuracy of the solution was tested by repeated calculation with the integration step halved.

The maximum error, Δ, of the calculated values $f_i''(0)$ shown in Table I together with the so far published data, is $5 \cdot 10^{i-10}$.*

The obtained solution enables calculation of the local and the mean friction coefficient c_f and $\bar{c}_f$ to be made. The local friction coefficient

$$c_f = 2\tau_w/\varrho U^2 = \pm 0{\cdot}5 \mathrm{Re}_x^{-0.5}[\partial^2 f(\xi, \eta)/\partial \eta^2]_{\eta=0} \qquad (19)$$

expressed in terms of the functions f_i is

$$c_f = \pm \mathrm{Re}_x^{-0.5} \sum_{i=0}^{3} 2^{2i-1} X^{0.5i} f_i''(0) \,. \qquad (20)$$

Similarly for the mean friction coefficient we have

$$\bar{c}_f \equiv (1/X) \int_0^X c_f \, \mathrm{d}X = \pm \mathrm{Re}_x^{-0.5} \sum_{i=0}^{3} 2^{2i}(i + 1)^{-1} X^{0.5i} f_i''(0) \,. \qquad (21)$$

The positive sign in Eqs (*19*) through (*21*) refers to the flow past a cylinder; the negative sign to the continuous cylinder.

Substituting the obtained values of $f_i''(0)$ in Eqs (*20*) and (*21*) the expressions for the friction coefficients take the form:

for the flow past a cylinder

$$c_f \mathrm{Re}_x^{0.5} = 0{\cdot}66411 + 1{\cdot}38864 X^{0.5} - 1{\cdot}31316 X + 3{\cdot}25069 X^{1.5} \,, \qquad (22)$$

$$\bar{c}_f \mathrm{Re}_x^{0.5} = 1{\cdot}32823 + 1{\cdot}38864 X^{0.5} - 0{\cdot}87544 X + 1{\cdot}62534 X^{1.5} \,; \qquad (23)$$

for the continuous cylinder

$$c_f \mathrm{Re}_x^{0.5} = 0{\cdot}88750 + 0{\cdot}76040 X^{0.5} - 0{\cdot}14830 X + 0{\cdot}07437 X^{1.5} \,, \qquad (24)$$

$$\bar{c}_f \mathrm{Re}^{0.5} = 1{\cdot}77499 + 0{\cdot}76040 X^{0.5} - 0{\cdot}09887 X + 0{\cdot}03718 X^{1.5} \,. \qquad (25)$$

In view of the used expansion the applicability of these equations is restricted to $X \leqq 0{\cdot}015$ for the flow past a cylinder where the fourth term of the series for the local coefficient amounts to 0·7% of the sum of the preceding terms and 30% of the third term. The applicability in case of the continuous cylinder, judging again from the magnitude of the fourth term (22% of the third term and 0·6% of the sum of the preceding terms) is restricted to $X \leqq 0{\cdot}2$.

* Detailed tables of $f_i^m(\eta)$ for $0 \leqq i \leqq 3$, the order of derivative $0 \leqq m \leqq 2$ and for $\eta \leqq 5{\cdot}6$ for the flow past a cylinder, or $\eta \leqq 18{\cdot}4$ for the continuous cylinder, will be made available upon request by the authors.

TABLE I

Values of $f''_i(0)$ for the Flow Past a Cylinder and a Continuous Cylinder

i	Flow past a cylinder			Continuous cylinder		
	this work	earlier data	reference	this work	earlier data	reference
0	1·328229345	1·32822932[a]	9	−1·774993253	−1·77500[a]	2
1	0·69432220	0·694322	10	−0·38019872	—	—
2	0·1641444	0·164144	10	0·0185379	—	—
3	0·101584	0·1016	11	−0·002324	—	—

[a] Owing to a different substitution the original paper reports the value of $f''_0(0)/4$.

TABLE II

Local Friction Coefficient for the Flow Past a Cylinder

X	$c_f \mathrm{Re}_x^{0\cdot5}$		$\frac{(28)-(22)}{(22)}$ %	X	$c_f \mathrm{Re}_x^{0\cdot5}$		$\frac{(28)-(22)}{(22)}$ %
	Eq. (22)	Eq. (28)			Eq. (22)	Eq. (28)	
0·00001	0·66849	0·58155	−13·00	0·001	0·70681	0·61815	−12·54
0·00005	0·67386	0·58670	−12·93	0·005	0·75689	0·66545	−12·08
0·0001	0·67787	0·59054	−12·88	0·01	0·79309	0·69911	−11·85
0·0005	0·69454	0·60646	−12·68	0·015	0·82046	0·72408	−11·75

A COMPARISON OF THE ACCURATE AND APPROXIMATE SOLUTIONS

An approximate solution of the boundary layer equations by the Kármán–Pohlhausen method has been performed for the flow past a cylinder by Glauert and Lighthill[14]; for the continuous cylinder, as has been mentioned, by Sakiadis[1]. The friction coefficient was expressed from these solutions as

$$c_f \mathrm{Re}_x^{0.5} = 2X^{0.5}/A\,, \tag{26}$$

where A is the inverse value of the dimensionless velocity gradient at the surface of the cylinder depending only on the parameter X

$$A = \pm(U/R)\,(\partial r/\partial u)_R\,, \tag{27}$$

with the positive (negative) sign for the flow past a cylinder (continuous cylinder).

A as a function of X may be computed from equations obtained by modification of those presented in the original papers:
Flow past a cylinder:

$$X = \sum_{n=1}^{\infty} 2^{n-1} A^{n+1} n^2/(n+1)\,(n+2)\,! \tag{28}$$

Continuous cylinder:

$$X = \sum_{n=1}^{\infty} 2^{n} A^{n+1} n/(n+1)\,(n+2)\,! \tag{29}$$

A comparison of the presented accurate solution of Eq. (9) for small X with the approximate solution for the flow past a cylinder and a continuous cylinder is given in Tables II and III.

Solution of the boundary layer on the continuous cylinder for large X has not been published to date. The values tabulated in Table IV were computed by the method due to Cebeci[13] worked out for the flow past a cylinder.*

CONCLUSION

The results for the flow past a cylinder give precision to the earlier analyses; it was found that the error of the approximate solution decreases with increasing X in the whole range of X, (refs[13–15]) even though the absolute values of the deviations reported by various authors are somewhat different as a consequence of the various methods of solving Eq. (9).

* The authors wish to thank Dr T. Cebeci for furnishing the results of numerical solution of Eq. (9) with the boundary conditions for the continuous cylinder[15].

TABLE III

Local Friction Coefficient for a Continuous Cylinder and $X \leqq 0{\cdot}2$

X	$c_f\,Re_x^{0\cdot5}$ Eq. (24)	$c_f\,Re_x^{0\cdot5}$ Eq. (29)	$\frac{(29)-(24)}{(24)}$ %	X	$c_f\,Re_x^{0\cdot5}$ Eq (24)	$c_f\,Re_x^{0\cdot5}$ Eq. (29)	$\frac{(29)-(24)}{(24)}$ %
0·0001	0·89509	0·82316	−8·04	0·01	0·96213	0·88237	−8·29
0·00025	0·89949	0·82702	−8·06	0·02	0·99228	0·90920	−8·37
0·001	0·91140	0·83750	−8·11	0·025	1·00432	0·91994	−8·40
0·00125	0·91420	0·83997	−8·12	0·1	1·11549	1·01976	−8·58
0·0025	0·92516	0·84963	−8·16	0·2	1·20459	1·10004	−8·68
0·005	0·94055	0·86324	−8·22	(0·25	1·23997	1·13183	−8·72)

TABLE IV

Local Friction Coefficient for a Continuous Cylinder and $X \leqq 250$

X	$c_f\,Re_x^{0\cdot5}$ Cebeci[15] a	$c_f\,Re_x^{0\cdot5}$ Eq. (29) b	$(b-a)/a$ %	X	$c_f\,Re_x^{0\cdot5}$ Cebeci[15] a	$c_f\,Re_x^{0\cdot5}$ Eq. (29) b	$(b-a)/a$ %
0·00025	0·8980	0·82702	−7·90	1	1·549	1·41967	−8·35
0·00125	0·9131	0·83997	−8·01	2·5	1·866	1·73112	−8·21
0·0025	0·9237	0·84963	−8·02	5	2·244	2·06136	−8·14
0·005	0·9389	0·86324	−8·06	10	2·721	2·50196	−8·05
0·01	0·9602	0·88237	−8·11	25	3·596	3·31547	−7·80
0·02	0·9903	0·90920	−8·19	50	4·546	4·16981	−8·28
0·025	1·002	0·91994	−8·19	100	5·798	5·30616	−8·48
0·1	1·113	1·01976	−8·38	200	7·458	6·81956	−8·56
0·25	1·236	1·13183	−8·43	250	8·125	7·40704	−8·84

For the continuous cylinder, however, the deviation of the approximate and the accurate solution in region $X \leqq 0{\cdot}2$ increases which is confirmed also by Cebeci's results[15] for small values of X ($X \leqq 0{\cdot}25$). In the region $0{\cdot}25 \leqq X \leqq 250$ the deviation of Cebeci's results[15] remains practically constant and its fluctuation is probably caused by the numerical method of solution.

The approximate solution of Sakiadis underestimates the friction factor for the continuous cylinder in region $X \leqq 250$ by about 8%. The assumption of Bourne and Elliston[5], that the error of Sakiadis' solution decreases with increasing X, thus is not correct. Nor are correct the results of Lee and Davis[7] according to which the Sakiadis' solution for small values of X is correct but for large X overestimates the friction factor. Solution of Vasudevan and Middleman[6] exhibiting marked deviations from Sakiadis' results in region of small X is based on false assumptions.[16]

To give precision to Cebeci's solution in region $0{\cdot}25 \leqq X \leqq 250$ and its extension to $X > 250$ is the purpose of the next paper.

LIST OF SYMBOLS

A	dimensionless inverse velocity gradient at the surface of cylinder (Eq. (*27*))
$c_f = 2\tau_w/U^2\varrho$	local friction coefficient
$\bar{c}_f$	mean friction coefficient
f	dimensionless stream function (Eq. (*8*) and (*12*))
r	radial coordinate originating on axis
R	radius of cylinder
$Re_R = RU/\nu$	Reynolds number based on radius
$Re_x = xU/\nu$	Reynolds number based on length
u	axial velocity component
U	speed of cylinder, or bulk fluid velocity
v	radial velocity component perpendicular to axis of cylinder
x	coordinate in direction of axis of cylinder
$X = Re_x/Re_R^2$	transformed coordinate ξ (Eq. (*6*))
η	transformed coordinate r (Eq. (*7*))
ν	kinematic viscosity
ξ	transformed coordinate x (Eq. (*6*))
ϱ	fluid density
τ_w	shear stress on the surface of cylinder
ψ	stream function

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